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On adjoint symmetry equations, integrating factors and solutions of nonlinear ODEs

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Abstract
We consider the role of the adjoint equation in determining explicit integrating factors and first integrals of nonlinear ODEs. In Chandrasekar *et al* (2006 *J. Math. Phys.* **47** 023508), the authors have used an extended version of the Prelle–Singer method for a class of nonlinear ODEs of the oscillator type. In particular, we show that their method actually involves finding a solution of the adjoint symmetry equation. Next, we consider a coupled second-order nonlinear ODE system and derive the corresponding coupled adjoint equations. We illustrate how the coupled adjoint equations can be solved to arrive at a first integral.

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1. Introduction

The study of nonlinear ordinary differential equations (ODEs) has been an ongoing endeavor for well over two centuries now, with significant contributions from many of the greatest mathematicians of all times such as Euler, Lie, Painlevé, Poincaré to mention just a few. Their contributions have ranged from finding explicit solutions of ODEs, to developing general methods of classifications, to a qualitative analysis of their solutions etc. These in turn have often led to the opening up of entirely new branches of study in algebra, topology, geometry and have shed new light on several physical phenomena.

Over the years many techniques have been developed to obtain exact solutions of various kinds of ODEs. However, there does not exist any single common method for obtaining their solutions. Nevertheless, the apparently different techniques share one common feature: they somehow tend to exploit the symmetries of ODEs. Consequently, symmetry analysis of ODEs has become one of the most powerful tools for analyzing them.The foundations of this method are contained in the works of Sophus Lie [1, 2].

It is also well known that the existence of a sufficient number of first integrals greatly simplifies the process of solving any ODE. Having said this, it is not always quite obvious what these first integrals are. Indeed, their determination is, in general, a non-trivial task. In the case of conservative mechanical systems, one often has just a single first integral—the energy. In this context, the semi-algorithmic procedure developed by Prelle and Singer deserves mention [4]. In its original version it applied to first-order ODEs involving rational functions with coefficients belonging to the field of complex numbers $\mathbb{C}$. Subsequently their method, which involved the use of Darboux polynomials, was extended by Singer to include Liouvillian first integrals [14], by Duarte *et al* [5, 6] and also by Man and MacCullum [13]. Chandrasekhar *et al* have also extended the analysis in a series of papers [7–9].

Even though systematic techniques for solving nonlinear ODEs can be traced to the seminal works of Lie, certain aspects of the subject appear to have lain dormant for over a century. Notable among these is the notion of their linearization. Of late it has received renewed attention and notable progress has been made in this regard. In fact, Chandrasekar *et al* have recently proposed an extended Prelle–Singer method, based on generalized transformations, to linearize a class of equations that cannot be linearized by invertible point transformations [7].

In this paper we show how the extended Prelle–Singer method as proposed by Chandrasekar *et al* may be incorporated into the existing adjoint symmetry equation method. Essentially, as their method deals with a pair of first-order equations, in the variables R and S (to be called the RS-pair), these can be combined to obtain the corresponding second-order adjoint symmetry equation.

It is natural to enquire if similar analogs/correspondences may be identified between the adjoint equation method and the RS-pair method for coupled second-order systems. The answer is affirmative. In fact, by using a coupled version of the adjoint symmetry equation, we derive the first integral for a relatively new system [12], which has appeared in connection with stellar dynamics.

This paper is organized as follows. In section 2 we recall certain standard results concerning the solution of ODEs by using first integrals, and introduce the linearized symmetry equation, for determining the Lie point symmetry generators. Section 3 reviews the extended Prelle–Singer method as outlined in [8, 9] and contains a derivation of the adjoint symmetry equation, based on this approach. We illustrate the relative advantages of these methods with a few simple examples. Section 4 is dedicated to coupled second-order ODEs.

2. Preliminaries

Consider an nth-order ODE in the normal form

$$y^{(n)} = w(x, y, y', \dots, y^{(n-1)}), \qquad \text{where} \quad y^{(k)} = \frac{\mathrm{d}^k y}{\mathrm{d}x^k}. \tag{2.1}$$

Corresponding to this ODE, there exists an equivalent first-order partial differential equation (PDE) in $(n + 1)$ variables [3, 10, 11],

$$\widetilde{D} f = (\partial_x + y'\partial_y + y''\partial_{y'} + \cdots + w\partial_{y^{(n-1)}}) f = 0, \tag{2.2}$$

in which the quantities $y', y''\dots$ are treated as independent variables at par with x, y.

Their equivalence is provided by the first integrals of (2.1). By definition a first integral is a global function $I = I(x, y, y', \dots, y^{(n-1)})$ that is constant along the solutions of (2.1), i.e.,

$$\frac{\mathrm{d}I}{\mathrm{d}x} = \widetilde{D}I = I_x + y'I_y + y''I_{y'} + \cdots + wI_{y^{(n-1)}} = 0. \tag{2.3}$$

Having determined a first integral, say $I = I(x, y, y', \dots, y^{(n-1)}) = I_0$, one can invert it to obtain

$$y^{(n-1)} = w_1(x, y, y', \dots, y^{(n-2)}; I_0)$$

provided $I_{y^{(n-1)}} \neq 0$. This shows that the existence of a first integral allows for the reduction in the order of the differential equation by 1. Furthermore, it is evident that every first integral is a solution of the linear PDE (2.2) and conversely.

Let us assume ϕ^α ($\alpha = 1, \dots, n$) denote a set of n functionally independent solutions of (2.1)/(2.2). Since each ϕ^α is a first integral, one has

$$\phi^\alpha(x, y, y', \dots, y^{(n-1)}) = I_0^\alpha, \qquad \alpha = 1, 2, \dots, n. \tag{2.4}$$

Consequently, by eliminating all derivatives from (2.4) one arrives at the general solution of (2.1) in the form

$$y = y\big(x; I_0^1, \dots, I_0^n\big),$$

the I_0^α's being essentially constants of integration.

As mentioned earlier, the determination of even a single first integral is in most cases a non-trivial task; hence while in principle the above procedure is fine, its practical application is often a daunting task, to say the least.

It is also well known that symmetries play a crucial role in the solutions of differential equations. In fact much of the existing literature on symmetries of ODEs is restricted to what are known as Lie point symmetries. The differential equation (2.1)/(2.2) is said to admit a Lie point symmetry with generator

$$\mathbf{X} = \xi(x, y)\partial_x + \eta(x, y)\partial_y + \eta^{(1)}\partial_{y'} + \cdots + \eta^{(k)}\partial_{y^{(k)}}, \qquad \text{where} \quad \eta^{(i)} = \frac{\mathrm{d}\eta^{(i-1)}}{\mathrm{d}x} - y^{(i)}\frac{\mathrm{d}\xi}{\mathrm{d}x},$$

if

$$[\mathbf{X}, \widetilde{D}] = g\widetilde{D} \tag{2.5}$$

holds. Here, $g = g(x, y, y', \dots, y^{(n-1)})$ is some function and $\eta^{(i)}$'s denote the prolongations of the vector field (infinitesimal generators) $\mathbf{X}^{(0)} = \xi(x, y)\partial_x + \eta(x, y)\partial_y$. For an nth-order ODE (2.1) the infinitesimal symmetry generators, when they exist, are determined from the linearized symmetry condition,

$$\eta^{(n)} = \xi w_x + \eta w_y + \eta^{(1)}w_{y'} + \cdots + \eta^{(n-1)}w_{y^{(n-1)}}, \tag{2.6}$$

when (2.1) holds [11]. In terms of the characteristic, $Q := \eta - y'\xi$, this condition may be written as

$$\widetilde{D}^n Q - w_{y^{(n-1)}}\widetilde{D}^{(n-1)}Q - \cdots - w_{y'}\widetilde{D}Q - w_y Q = 0. \tag{2.7}$$

For example when $y'' = w(x, y, y')$, the linearized symmetry condition is a second-order linear PDE

$$\widetilde{D}^2 Q - w_{y'}\widetilde{D}Q - w_y Q = 0 \tag{2.8}$$

with vector field

$$\widetilde{D} = \partial_x + y'\partial_y + w(x, y, y')\partial_{y'}.$$

3. Adjoint symmetries and integrating factors

The following equation is known as the adjoint of the linearized symmetry condition (2.7), and its solutions are usually called the adjoint symmetries

$$\widetilde{D}^n\Lambda + \widetilde{D}^{n-1}(w_{y^{(n-1)}}\Lambda) - \widetilde{D}^{n-2}(w_{y^{(n-2)}}\Lambda) + \cdots + (-1)^{n-1}w_y\Lambda = 0. \tag{3.1}$$

It must be stressed however that these solutions are neither symmetries nor generators of symmetries, and it is more appropriate to call a solution a *cocharacteristic* [11]. A systematic procedure for finding the solutions of (3.1) is to use an ansatz for Λ, for example, to assume that they are independent of $y^{(n-1)}$ or to even assume a suitable rational structure.

3.1. Review of the extended Prelle–Singer method

Let us consider once again the equation

$$y^{(n)} = w(x, y, y', \ldots, y^{(n-1)}), \tag{3.2}$$

together with the base one-forms $\mathrm{d}x, (\mathrm{d}y - y'\,\mathrm{d}x), \ldots, (\mathrm{d}y^{(n-1)} - w\mathrm{d}x)$. The null form obtained by multiplying all but the first one-form by functions $S_i(x, y, y', \ldots, y^{(n-1)})$ where $i = 0, \ldots, n-1$ and demanding that after addition the resultant one-form be exact is

$$\begin{aligned}-(S_0y' + S_1y'' + \cdots + S_{n-2}y^{(n-1)} + S_{n-1}w)\,\mathrm{d}x\\ + (S_0\,\mathrm{d}y + S_1\,\mathrm{d}y' + \cdots + S_{n-2}\,\mathrm{d}y^{(n-2)} + S_{n-1}\,\mathrm{d}y^{(n-1)})\\ = \mathrm{d}I(x, y, y', \ldots, y^{(n-1)}) = 0.\end{aligned} \tag{3.3}$$

This implies

$$I_x = -(S_0y' + S_1y'' + \cdots + S_{n-2}y^{(n-2)} + wS_{n-1}) \tag{3.4}$$

$$I_y = S_0, \qquad I_{y'} = S_1, \ldots, I_{y^{(n-1)}} = S_{n-1}. \tag{3.5}$$

Clearly I is a first integral of the equation (3.2), provided it satisfies the integrability criteria

$$I_{xy^{(j)}} = I_{y^{(j)}x}, \qquad j = 0, \ldots, n-1, \tag{3.6}$$

$$I_{y^{(j)}y^{(k)}} = I_{y^{(k)}y^{(j)}}, \qquad 0 \leqslant j < k \leqslant n-1. \tag{3.7}$$

The vector field associated with (3.2) is

$$\widetilde{D} = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + \cdots + w\frac{\partial}{\partial y^{(n-1)}}, \tag{3.8}$$

in terms of which the integrability conditions (3.6) may be expressed as follows:

$$-\widetilde{D}[S_{n-1}] = (w_{y^{(n-1)}}S_{n-1} + S_{n-2}) \tag{3.9}$$

$$-\widetilde{D}[S_{n-2}] = (w_{y^{(n-2)}}S_{n-1} + S_{n-3}) \tag{3.10}$$

$$\vdots$$

$$-\widetilde{D}[S_2] = (w_{y'}S_{n-1} + S_0) \tag{3.11}$$

$$-\widetilde{D}[S_0] = w_yS_{n-1}. \tag{3.12}$$

The remaining integrability conditions (3.7) are all satisfied if

$$\frac{\partial S_{n-1}}{\partial y^{(j)}} = \frac{\partial S_j}{\partial y^{(n-1)}}, \qquad 0 \leqslant j \leqslant n-2. \tag{3.13}$$

Our primary interest is to know S_{n-1}, since the remaining ones can be determined algebraically from (3.9)–(3.12) in a recursive manner. Eliminating the S_i's by successively applying the vector field $\widetilde{D}$ to (3.9) and using the remaining ones, we obtain finally

$$\widetilde{D}^n[S_{n-1}] + \widetilde{D}^{n-1}[w_{y^{(n-1)}} S_{n-1}] - \widetilde{D}^{n-2}[w_{y^{(n-2)}} S_{n-1}] + \cdots + (-1)^{n-1} w_y S_{n-1} = 0. \tag{3.14}$$

But this is precisely the adjoint equation corresponding to the linearized symmetry equation (3.1), [11]. Thus the integrating factors of (2.1) are just the solutions of (3.14), which fulfil the integrability criteria stated in (3.13). Consequently, determination of the integrating factor S_{n-1} of (3.2) is basically equivalent to finding a solution of this equation. (The connection with the notation used in [9] is established by the following substitutions: $S_j \longrightarrow RS_{j+1}, \forall j = 0, \ldots, n-3$ and $S_{n-1} \longrightarrow R$.). The usual procedure to tackle such PDEs is to make an ansatz for S_{n-1}, for example assuming it to be a polynomial in $y^{(n-1)}$ of some suitable degree, and then obtaining its coefficients in a recursive manner. In their works, Chandrasekar *et al* have made a very interesting ansatz, in which they assumed a rational form for S_{n-1}. As a consequence, instead of solving the adjoint equation directly, they solved the set (3.9)–(3.12) of first-order equations by making appropriate ansätze for the S_i's. Suppose Λ^i be the solution(s) of the adjoint equation. Setting $S_{n-1} = \Lambda^i$ one can calculate the remaining S_j's in a recursive manner and check if (3.13) holds. In the event such an integrating factor exists and satisfies the integrability condition, its associated first integral may be obtained from the relation

$$I^i = \int S_0^i(\mathrm{d}y - y'\,\mathrm{d}x) + S_1^i(\mathrm{d}y' - y''\,\mathrm{d}x) + \cdots S_{n-1}^i(\mathrm{d}y^{(n-1)} - w\,\mathrm{d}x). \tag{3.15}$$

Essentially, therefore, one can choose to either solve the adjoint equation directly and obtain S_{n-1} through some suitable ansätze or make suitable ansätze for the S_k's and solve a set of n first-order PDEs. In general the former involves solving a single higher order equation, while the latter involves solving a system of first-order linear PDEs. It appears from the works [7–9] that the latter is much easier to implement, as far as practical computations are concerned. In the following, we illustrate these points with examples of second-order equations.

3.2. Some illustrative examples

Example 1. $y'' = w(x, y, y') = \frac{3y'^2}{y} + \frac{y'}{x}$.

Here the system of coupled first-order PDEs for the unknown functions S_0, S_1 is:

$$\widetilde{D}S_1 = -(w_{y'}S_1 + S_0) \tag{3.16}$$

$$\widetilde{D}S_0 = -w_y S_1, \tag{3.17}$$

where $\widetilde{D} = \partial_x + y'\partial_y + w\partial_{y'}$; the integrability condition is simply

$$S_{1y} = S_{0y'}. \tag{3.18}$$

The adjoint equation is

$$\widetilde{D}^2 S_1 + \widetilde{D}(w_{y'}S_1) - w_y S_1 = 0. \tag{3.19}$$

Assuming $\Lambda = S_1$ to be a solution of (3.19) independent of y', we have upon equating the coefficients of different powers of y' the following set of equations:

$$15\Lambda + 9y\Lambda_y + y^2\Lambda_{yy} = 0$$
$$3\Lambda + 3x\Lambda_x + y\Lambda_y + xy\Lambda_{xy} = 0$$
$$-\Lambda + x\Lambda_x + x^2\Lambda_{xx} = 0.$$

Their structure suggests an ansatz of the form $\Lambda = x^\alpha y^\beta$. One can verify that this leads to three solutions, namely,

$$\Lambda^1(x, y) = \frac{x}{y^3}, \qquad \Lambda^2(x, y) = \frac{1}{xy^3} \qquad \text{and} \qquad \Lambda^3(x, y) = \frac{1}{xy^5}.$$

However, only Λ^1 and Λ^2 are acceptable, as the other does not satisfy the integrability criterion. The results are summarized below along with the respective first integrals:

$$(i) \quad \Lambda^1 = S_1^1 = \frac{x}{y^3}, \qquad S_0^1 = -\frac{x}{y^3}\left(\frac{2}{x} + \frac{3y'}{y}\right), \qquad \text{with} \quad I^1(x, y, y') = \frac{xy' + y}{y^3}$$

$$(ii) \quad \Lambda^2 = S_1^2 = \frac{1}{xy^3}, \qquad S_0^2 = -\frac{3y'}{xy^4}, \qquad \text{with} \quad I^2(x, y, y') = \frac{y'}{xy^3}.$$

The first integral I^2 was obtained by Duarte *et al* in [5]. But for some reason the other one was not mentioned.

Example 2. In this example we study the equation

$$y'' = w(x, y, y') = -(kyy' + \lambda y),$$

where k, λ are constants, which represents a damped harmonic oscillator. As before one has to solve the adjoint symmetry equation (3.1) for $n = 2$, namely,

$$(w_{xy'} + y'w_{yy'} + ww_{y'y'} - w_y)\Lambda + w_{y'}\Lambda_x + (w + y'w_{y'})\Lambda_y + (w_x + 2ww_{y'} + y'w_y)\Lambda_{y'}$$
$$+ \Lambda_{xx} + 2y'\Lambda_{xy} + y'^2\Lambda_{yy} + 2w\Lambda_{xy'} + 2wy'\Lambda_{yy'} + w^2\Lambda_{y'y'} = 0.$$

Solving this PDE is a rather daunting task even when $w(x, y, y')$ is fairly simple. It is therefore natural to make certain simplifying assumptions regarding the functional dependence of Λ. For instance one can begin by assuming Λ to be independent of a particular variable, say x, and see if that leads to a more manageable form of the adjoint equation. Alternatively, one may at the very outset assume that Λ depends on any one of the three variables x, y or y'. The choice of procedure to be adopted is one of sheer convenience. We illustrate this by first making the simplifying assumption $\Lambda_x = 0$, which leads to

$$(w_{xy'} + y'w_{yy'} + ww_{y'y'} - w_y)\Lambda + (w + y'w_{y'})\Lambda_y + (w_x + 2ww_{y'} + y'w_y)\Lambda_{y'}$$
$$+ y'^2\Lambda_{yy} + 2wy'\Lambda_{yy'} + w^2\Lambda_{y'y'} = 0.$$

This is a linear parabolic PDE. Since $w = -(kyy' + \lambda y)$ we have

$$w_x = w_{y'y'} = 0, \qquad w_{y'} = -ky, \qquad w_y = -(ky' + \lambda) \qquad \text{and} \qquad w_{yy'} = -k.$$

As solving this PDE is still rather formidable, let us further assume $\Lambda_y = 0$. In other words Λ is just a function of y' and our equation simplifies further to

$$(w_{xy'} + y'w_{yy'} + ww_{y'y'} - w_y)\Lambda + (w_x + 2ww_{y'} + y'w_{y'})\Lambda_{y'} + w^2\Lambda_{y'y'} = 0.$$

Plugging in the expressions for partial derivatives of w and equating the coefficients of different powers of y then leads to the following set of equations:

$$(ky' + \lambda)y'\Lambda_{y'} = \lambda\Lambda$$
$$2k\Lambda_{y'} + (ky' + \lambda)\Lambda_{y'y'} = 0.$$

These equations admit the particular solution $\Lambda^1(y') = \frac{y'}{(ky'+\lambda)}$ and one finds with $S_1^1 = \Lambda^1 = \frac{y'}{(ky'+\lambda)}$ that $S_0^1 = y$. The integrability condition $S_{1y}^1 = S_{0y'}^1$ is trivially satisfied and the corresponding first integral is

$$I^1(x, y, y') = y' + \frac{1}{2}ky^2 - \frac{\lambda}{k}\log(ky' + \lambda).$$

Note that this first integral is independent of x by construction. For such first integrals, the method devised by Chandrasekar *et al* allows us to determine the form of S_0 *a priori*. We dwell on this aspect in the following section.

3.3. First integrals independent of a particular coordinate

In this subsection, we shall discuss the issue of first integrals independent of a particular coordinate. This usually leads to a reduction of the order of the equation, as will be explained below. The general ideas contained here will be illustrated with a specific example of a generic second-order ODE of the Liénard type.

An interesting feature occurs when the first integral is independent of a particular variable, say x, i.e., $I_x = 0$. Then, in general, (3.4) implies

$$S_0 = -\frac{1}{y'}(y''S_1 + \cdots + S_{n-2}y^{(n-1)} + S_{n-1}w),$$

which enables us to eliminate S_0, and causes a reduction in the order of the equations for determining the integrating factor. For instance in the case of a second-order ODE, we have $S_0y' + wS_1 = 0$, leading to $S_0 = -\frac{w}{y'}S_1$. As a result, one is left with a first-order PDE for determining S_1, namely,

$$\widetilde{D}(S_1) = -\left(w_{y'} - \frac{w}{y}\right)S_1. \tag{3.20}$$

On the other hand, for a third-order equation, we have

$$S_0 = -\frac{y''S_1 + wS_2}{y'}.$$

Elimination of S_0 from the system of equations (3.9)–(3.12) with $n = 3$ then requires us to solve for S_1 and S_2 from the coupled system:

$$\widetilde{D}[S_2] = -(w_{y''}S_2 + S_1)$$
$$\widetilde{D}[S_1] = -\left(\left(w_{y'} - \frac{w}{y'}\right)S_2 - \frac{y''}{y'}S_1\right).$$

This in turn leads to the following second-order equation for the integrating factor S_2:

$$\widetilde{D}^2S_2 + \widetilde{D}(w_{y''}S_2) - \frac{y''}{y'}\widetilde{D}S_2 - \left\{\left(w_{y'} - \frac{w}{y'}\right) + \frac{y''}{y'}w_{y''}\right\}S_2 = 0. \tag{3.21}$$

Thus the absence of one 'coordinate' in a first integral causes only marginal simplification, namely a reduction, by one, in the order of the equation to be solved for the integrating factor. Nevertheless this is extremely useful for second-order equations $y'' = w(x, y, y')$, since one is then required to solve a *single* first-order linear PDE for the integrating factor S_1. This fact was exploited in [7, 8]. Although in general for $n \geqslant 3$, the existence of an x independent first integral may not always lead to a substantial reduction of computational labor; nevertheless it is instructive to look into the *RS* method more carefully, as it has proved to be immensely successful in determining first integrals of many highly nonlinear oscillator-type systems. Generally, for equations of the generic form $y'' = -f_1(y)y' - f_0(y)$, (3.20) reduces to

$$\widetilde{D}S_1 = -\frac{f_0(y)}{y'}S_1.$$

The solution S_1^1 of example 2 suggests the ansatz $S_1 = \frac{y'}{h(y,y')}$ with the consequence

$$\widetilde{D}S_1 = \frac{\widetilde{D}(y')}{h} - \frac{y'}{h}\widetilde{D}h = -\frac{f_0(y)}{h}.$$

Therefore, the problem now reduces to a determination of the function $h(y, y')$ from the following relation (since $\widetilde{D}(y') = w$):

$$\begin{aligned} y'\widetilde{D}(h) &= (w + f_0)h = -f_1(y)y'h \\ \widetilde{D}(h) &= -f_1(y)h. \end{aligned} \tag{3.22}$$

The resulting PDE for h is explicitly given by

$$y'h_y + (-f_1y' - f_0)h_{y'} = -f_1y'h.$$

For $f_1 = ky$ and $f_0 = \lambda y$, assuming furthermore that h is independent of y, we obtain $h(y') = C(ky' + \lambda)$. Thus once again we get the solutions, setting constant $C = 1$,

$$S_1 = \frac{y'}{(ky' + \lambda)} \qquad \text{and} \qquad S_0 = y,$$

which satisfy the integrability criterion.

As pointed out in [9], it is often more convenient to modify the ansatz for S_1 to $S_1 = \frac{y'}{h(y,y')^r}$ to handle more complicated situations.

For generic equations of the form (Liénard type)

$$y'' = -f_1(y)y' - f_0(y)$$

with this ansatz for S_1, (3.22) is modified to

$$r\widetilde{D}(h) = -f_1(y)h. \tag{3.23}$$

Assuming $h(y, y') = A(y) + B(y)y' + C(y)y'^2$, substitution into (3.23) leads to the following set of equations for determining the unknown functions A, B, C upon equating coefficients of different powers of y':

$$C_y = 0, \qquad rB_y = (2rf_0 - f_1)C, \qquad rA_y = (rf_0 - f_1)B - 2rCf_1 \qquad \text{and} \qquad rf_0B = f_1A. \tag{3.24}$$

Suppose

$$f_0(y) = \lambda y^{\xi} \qquad \text{and} \qquad f_1(y) = \mu y^{\eta},$$

where λ, μ are parameters and ξ, η are constants. We obtain the following solutions for C, B and A:

$$C(y) = \gamma, \qquad rB(y) = \mu\gamma\frac{(2r-1)}{\eta+1}y^{\eta+1} + \beta$$

$$rA(y) = \frac{2\lambda r\gamma}{\xi+1}y^{\xi+1} + \mu(r-1)\left[\frac{(2r-1)\mu\gamma}{2r(\eta+1)^2}y^{2(\eta+1)} + \frac{\beta}{r(\eta+1)}y^{\eta+1}\right] + \alpha.$$

Here α, β and γ are constants of integration. From the last condition in (3.24), i.e., $rf_0B = f_1A$, it follows, assuming $\xi \neq \eta$, that $\alpha = \beta = 0$ and leads to the following relation:

$$\lambda r\left[\frac{(2r-1)}{(\eta+1)} - \frac{2}{(\xi+1)}\right]y^{\xi+\eta+1} = \frac{\mu^2(r-1)(2r-1)}{2r(\eta+1)^2}y^{3\eta+2}. \tag{3.25}$$

One can then identify two possible cases.

(a) When $r = 1$ we have $\xi = 2\eta + 1$ and $A(y) = \frac{\lambda\gamma}{(\eta+1)} y^{2(\eta+1)}$ and $B(y) = \frac{\mu\gamma}{(\eta+1)} y^{(\eta+1)}$. The corresponding integrating factor is

$$S_1^a = \frac{y'}{\left[\frac{\lambda\gamma}{(\eta+1)} y^{2(\eta+1)} + \frac{\mu\gamma}{(\eta+1)} y^{(\eta+1)} y' + \gamma y'^2\right]} \qquad \text{and} \qquad S_0^a = \frac{\mu y^\eta y' + \lambda y^{2\eta+1}}{y'} S_1.$$

(b) For $r \neq 1$, assuming the exponents of y in (3.25) to be equal, we find once again $\xi = 2\eta + 1$. Upon equating their coefficients, we obtain a quadratic equation for the exponent r, occurring in the denominator of the integrating factor, with solution $r = \frac{\mu^2}{4\lambda(\eta+1)}\left[1 \pm \sqrt{1 - \frac{4\lambda}{\mu^2}(\eta + 1)}\right]$. Therefore, in this case $S_1^b = \frac{y'}{h^r}$ where

$$h(y, y') = \frac{\gamma}{(\eta + 1)}\left[\lambda + \mu^2 \frac{(r - 1)(2r - 1)}{2r^2(\eta + 1)}\right] y^{2(\eta+1)} + \frac{\gamma\mu(2r - 1)}{r(\eta + 1)} y^{\eta+1} y' + \gamma y'^2.$$

4. Coupled second-order equations

In this section, we consider a system of second-order ODEs to illustrate an application of the coupled version of the adjoint equation.

Let us consider the system of coupled second-order equations:

$$\ddot{x} = \phi_1(x, y) \qquad \text{and} \qquad \ddot{y} = \phi_2(x, y). \tag{4.1}$$

As before, consider the following base one forms $(\mathrm{d}x - \dot{x}\,\mathrm{d}t)$, $(\mathrm{d}y - \dot{y}\,\mathrm{d}t)$, $(\mathrm{d}\dot{x} - \phi_1\,\mathrm{d}t)$, $(\mathrm{d}\dot{y} - \phi_2\,\mathrm{d}t)$. Let S_1, S_2 and R_1, R_2 be functions such that

$$S_1(\mathrm{d}x - \dot{x}\,\mathrm{d}t) + S_2(\mathrm{d}y - \dot{y}\,\mathrm{d}t) + R_1(\mathrm{d}\dot{x} - \phi_1\,\mathrm{d}t) + R_2(\mathrm{d}\dot{y} - \phi_2\,\mathrm{d}t) = \mathrm{d}I(t, x, y, \dot{x}, \dot{y}) = 0. \tag{4.2}$$

Hence

$$I_t = -(S_1\dot{x} + S_2\dot{y} + R_1\phi_1 + R_2\phi_2) \tag{4.3}$$

$$I_x = S_1, \quad I_y = S_2, \quad I_{\dot{x}} = R_1, \quad I_{\dot{y}} = R_2. \tag{4.4}$$

The functions R_1, R_2 are the integrating factors. Compatibility of the set of (4.3) and (4.4), namely,

$$\begin{array}{lllll} I_{tx} = I_{xt}, & I_{ty} = I_{yt}, & I_{t\dot{x}} = I_{\dot{x}t}, & I_{t\dot{y}} = I_{\dot{y}t} & \\ I_{xy} = I_{yx}, & I_{x\dot{x}} = I_{\dot{x}x}, & I_{x\dot{y}} = I_{\dot{y}x}, & I_{y\dot{x}} = I_{\dot{x}y}, & I_{y\dot{y}} = I_{\dot{y}y}, \end{array} \tag{4.5}$$

requires that the following hold:

$$D[R_1] = -(S_1 + R_1\phi_{1\dot{x}} + R_2\phi_{2\dot{x}}) \tag{4.6}$$

$$D[R_2] = -(S_2 + R_1\phi_{1\dot{y}} + R_2\phi_{2\dot{y}}) \tag{4.7}$$

$$D[S_1] = -(R_1\phi_{1x} + R_2\phi_{2x}) \tag{4.8}$$

$$D[S_2] = -(R_1\phi_{1y} + R_2\phi_{2y}), \tag{4.9}$$

where $D = \partial_t + \dot{x}\partial_x + \dot{y}\partial_y + \phi_1\partial_{\dot{x}} + \phi_2\partial_{\dot{y}}$. It is evident that once R_1, R_2 are known the remaining S_1, S_2 can be determined algebraically from (4.6) and (4.7). Since our basic aim is to determine the integrating factors, we can eliminate, say, S_1 by differentiating (4.6) and using (4.8) to obtain

$$D^2[R_1] + D[R_1\phi_{1\dot{x}} + R_2\phi_{2\dot{x}}] - (R_1\phi_{1x} + R_2\phi_{2x}) = 0. \tag{4.10}$$

Similarly eliminating S_2 yields

$$D^2[R_2] + D[R_1\phi_{1\dot{y}} + R_2\phi_{2\dot{y}}] - (R_1\phi_{1y} + R_2\phi_{2y}) = 0. \tag{4.11}$$

Equations (4.10)–(4.11) constitute the coupled version of the adjoint equation (3.1) when $n = 2$.

One needs to check, of course, that the solutions of the coupled adjoint equations indeed satisfy the compatibility conditions (4.5). In general one employs an ansatz for R_1, R_2 in order to solve the system of PDEs (4.10)–(4.11). From a knowledge of R_1, R_2 and S_1, S_2 it is straightforward to obtain the first integral from

$$I = \int S_1(\mathrm{d}x - \dot{x}\,\mathrm{d}t) + S_2(\mathrm{d}y - \dot{y}\,\mathrm{d}t) + R_1(\mathrm{d}\dot{x} - \phi_1\,\mathrm{d}t) + R_2(\mathrm{d}\dot{y} - \phi_2\,\mathrm{d}t). \tag{4.12}$$

Example 3. Consider the following system of second-order equations:

$$\begin{aligned} &\ddot{x} + \frac{\alpha}{x^2}g(u) - \frac{\lambda}{x^3} = 0 \\ &\ddot{y} + \frac{\beta}{x^2}f(u) - \frac{\mu}{y^3} = 0, \qquad u = \frac{y}{x}. \end{aligned} \tag{4.13}$$

Here α, β, λ and μ are parameters and f and g are arbitrary functions. Writing these equations in the form $\ddot{x} = \phi_1(x, y)$ and $\ddot{y} = \phi_2(x, y)$, we identify

$$\phi_1(x, y) = -\frac{\alpha}{x^2}g(u) + \frac{\lambda}{x^3} \qquad \text{and} \qquad \phi_2(x, y) = -\frac{\beta}{x^2}f(u) + \frac{\mu}{y^3}.$$

Note here ϕ_1 and ϕ_2 are velocity independent and for a time-independent first integral $I_t = 0$, we may take $D = \dot{x}\partial_x + \dot{y}\partial_y + \phi_1\partial_{\dot{x}} + \phi_2\partial_{\dot{y}}$. In that event, with the following ansatz for R_1 and R_2, namely,

$$R_1 = a_1(x, y)\dot{x} + a_2(x, y)\dot{y} \qquad \text{and} \qquad R_2 = b_1(x, y)\dot{x} + b_2(x, y)\dot{y}, \tag{4.14}$$

(4.10) and (4.11) yield the following equations:

$$\begin{aligned} &\dot{x}^3a_{1xx} + \dot{x}^2\dot{y}(a_{2xx} + 2a_{1xy}) + \dot{x}\dot{y}^2(2a_{2xy} + a_{1yy}) + a_{2yy}\dot{y}^3 \\ &\qquad + \dot{x}\{(\phi_1a_1 + \phi_2a_2)_x + 2a_{1x}\phi_1 + (a_{2x} + a_{1y})\phi_2\} \\ &\qquad + \dot{y}\{(\phi_1a_1 + \phi_2a_2)_y + 2a_{2y}\phi_2 + (a_{2x} + a_{1y})\phi_1\} \\ &\quad = \dot{x}(\phi_{1x}a_1 + \phi_{2x}b_1) + \dot{y}(\phi_{1x}a_2 + \phi_{2x}b_2), \end{aligned} \tag{4.15}$$

$$\begin{aligned} &\dot{x}^3b_{1xx} + \dot{x}^2\dot{y}(b_{2xx} + 2b_{1xy}) + \dot{x}\dot{y}^2(2b_{2xy} + b_{1yy}) + b_{2yy}\dot{y}^3 \\ &\qquad + \dot{x}\{(\phi_1b_1 + \phi_2b_2)_x + 2b_{1x}\phi_1 + (b_{2x} + b_{1y})\phi_2\} \\ &\qquad + \dot{y}\{(\phi_1b_1 + \phi_2b_2)_y + 2b_{2y}\phi_2 + (b_{2x} + b_{1y})\phi_1\} \\ &\quad = \dot{x}(\phi_{1y}a_1 + \phi_{2y}b_1) + \dot{y}(\phi_{1y}a_2 + \phi_{2y}b_2). \end{aligned} \tag{4.16}$$

Equating coefficients of different powers of the velocities we obtain the following system of equations:

$$a_{1xx} = 0, \qquad a_{2xx} + 2a_{1xy} = 0, \qquad a_{1yy} + 2a_{2xy} = 0, \qquad a_{2yy} = 0, \tag{4.17}$$

$$(\phi_1a_1 + \phi_2a_2)_x + 2a_{1x}\phi_1 + (a_{2x} + a_{1y})\phi_2 = (\phi_{1x}a_1 + \phi_{2x}b_1), \tag{4.18}$$

$$(\phi_1a_1 + \phi_2a_2)_y + 2a_{2y}\phi_2 + (a_{2x} + a_{1y})\phi_1 = (\phi_{1x}a_2 + \phi_{2x}b_2) \tag{4.19}$$

$$b_{1xx} = 0, \qquad b_{2xx} + 2b_{1xy} = 0, \qquad b_{1yy} + 2b_{2xy} = 0, \qquad a_{2yy} = 0, \tag{4.20}$$

$$(\phi_1 b_1 + \phi_2 b_2)_x + 2b_{1x}\phi_1 + (b_{2x} + b_{1y})\phi_2 = (\phi_{1y}a_1 + \phi_{2y}b_1), \tag{4.21}$$

$$(\phi_1 b_1 + \phi_2 b_2)_y + 2b_{2y}\phi_2 + (b_{2x} + b_{1y})\phi_1 = (\phi_{1y}a_2 + \phi_{2y}b_2). \tag{4.22}$$

Observe that the choice a_k = constant and b_k = constant $(k = 1, 2)$ satisfies (4.17) and (4.20), while the remaining equations then simplify to

$$\phi_{2x}(b_1 - a_2) = 0, \qquad \phi_{1y}(a_2 - b_1) = 0$$

$$(\phi_{1x} - \phi_{2y})a_2 - \phi_{1y}a_1 + \phi_{2x}b_2 = 0$$

$$\phi_{1y}a_1 + (\phi_{2y} - \phi_{1x})b_1 - \phi_{2x}b_2 = 0.$$

The first two equations imply $a_2 = b_1$, which renders the second and the third equations identical, namely,

$$(\phi_{1x} - \phi_{2y})a_2 - \phi_{1y}a_1 + \phi_{2x}b_2 = 0.$$

If equations (4.13) are derivable from a potential then it is necessary that $\phi_{1y} = \phi_{2x}$. With this in mind the above equation can be satisfied by making the choice $a_2 = b_1 = 0$ whilst a_1 and b_2 are arbitrary. Therefore, the choice $a_1 = b_2 = 1$ and $a_1 = b_1 = 0$ leads to the following solution:

$$R_1 = \dot{x} \qquad R_2 = \dot{y}. \tag{4.23}$$

In this case the solutions of S_1 and S_2 from (4.6) and (4.7) are found to be

$$S_1 = -\phi_1 = \frac{\alpha}{x^2}g(u) - \frac{\lambda}{x^3}$$

$$S_2 = -\phi_2 = \frac{\beta}{x^2}f(u) - \frac{\mu}{y^3}, \qquad u = \frac{y}{x}.$$

Using the above values of R_i and S_i $(i = 1, 2)$ we obtain from (4.12) the first integral as

$$I(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{\lambda}{2x^2} + \frac{\mu}{2y^2} + N(x, y),$$

where

$$N(x, y) = \int \frac{\alpha}{x^2}g(u)\,\mathrm{d}x + \int \frac{\beta}{x^2}f(u)\,\mathrm{d}y.$$

On the other hand the condition $\phi_{1y} = \phi_{2x}$ translates to

$$\alpha g'(u) + 2\beta f(u) + \beta u f'(u) = 0. \tag{4.24}$$

Using this condition $N(x, y)$ may be evaluated and we find that

$$N(x, y) = -\frac{\beta}{x}\left(\frac{\alpha}{\beta}g(u) + uf(u)\right).$$

Hence a first integral for the system of second-order equations is

$$I(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{\lambda}{2x^2} + \frac{\mu}{2y^2} - \frac{\beta}{x}\left(\frac{\alpha}{\beta}g(u) + uf(u)\right). \tag{4.25}$$

Let us now look for another solution set of the coupled adjoint equations for R_1 and R_2. It is easily verified that

$$a_1(x, y) = y^2, \qquad a_2(x, y) = -xy = b_1(x, y) \qquad \text{and} \qquad b_2(x, y) = x^2 \tag{4.26}$$

satisfy (4.17) and (4.20) while (4.18) and (4.22) are identically satisfied. The remaining equations (4.19) and (4.21) become identical and reduce to the following equation:

$$3(y\phi_1 - x\phi_2) = (\phi_{2y} - \phi_{1x})xy - \phi_{1y}y^2 + \phi_{2x}x^2. \tag{4.27}$$

Substituting the values of ϕ_i $(i = 1, 2)$ and their derivatives leads to the following condition on the functions f and g, namely:

$$\alpha u g(u) - \beta f(u) = 0, \qquad u = \frac{y}{x}. \tag{4.28}$$

From (4.26) we derive the following solution for R_i $(i = 1, 2)$:

$$R_1 = -y(x\dot{y} - y\dot{x}) \qquad \text{and} \qquad R_2 = x(x\dot{y} - y\dot{x}). \tag{4.29}$$

The corresponding values of S_i $(i = 1, 2)$ are now

$$S_1 = (x\dot{y} - y\dot{x})\dot{y} - \lambda\frac{y^2}{x^3} + \mu\frac{x}{y^2} \qquad \text{and} \qquad S_2 = -(x\dot{y} - y\dot{x})\dot{x} + \lambda\frac{y}{x^3} - \mu\frac{x^2}{y^3}, \tag{4.30}$$

where use has been made of the condition (4.28). Hence from (4.12) we obtain another first integral given by

$$I(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(y\dot{x} - x\dot{y})^2 + \frac{\lambda}{2}\left(\frac{y}{x}\right)^2 + \frac{\mu}{2}\left(\frac{x}{y}\right)^2. \tag{4.31}$$

The two first integrals given by (4.25) and (4.31) will be valid simultaneously provided we can find functions f and g which satisfy (4.24) and (4.28). It is easily verified that these require the functions f and g to be given by

$$g(u) = \frac{1}{(1 + u^2)^{3/2}} \qquad \text{and} \qquad f(u) = \frac{\alpha}{\beta}\frac{u}{(1 + u^2)^{3/2}},$$

respectively. Under the circumstances the system of second-order equations reduces to the following well-known system

$$\ddot{x} + \frac{\alpha x}{(x^2 + y^2)^{3/2}} - \frac{\lambda}{x^3} = 0 \qquad \ddot{y} + \frac{\alpha y}{(x^2 + y^2)^{3/2}} - \frac{\mu}{y^3} = 0,$$

with the first integrals

$$I_1 = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{\lambda}{2x^2} + \frac{\mu}{2y^2} - \frac{\alpha}{\sqrt{x^2 + y^2}}$$

$$I_2 = \frac{1}{2}(y\dot{x} - x\dot{y})^2 + \frac{\lambda}{2}\left(\frac{y}{x}\right)^2 + \frac{\mu}{2}\left(\frac{x}{y}\right)^2.$$

A more interesting situation from the physical point of view arises when the functions f and g satisfy condition (4.24) but *not* condition (4.28). In that event the system of equations (4.13) admits just one first integral given by (4.25), with f and g satisfying (4.24). In [12] the authors obtained a system of equations similar in structure to (4.13), in the context of the dynamics of stellar systems, with

$$f(u) = 2(1 - ug(u)).$$

Condition (4.24) then leads to the following differential equation determining $g(u)$:

$$(1 - 2u^2)g'(u) = 2(3ug(u) - 2)$$

and the first integral assumes the form (setting all the parameters equal to unity)

$$I(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(y\dot{x} - x\dot{y})^2 + \frac{1}{2x^2} + \frac{1}{2y^2} - \frac{1}{x}(2u + (1 - 2u^2)g(u)), \qquad u = \frac{y}{x}.$$

In fact this first integral serves as the Hamiltonian.

5. Outlook

In this paper we have studied the *RS*-pair method, for determination of first integrals of ODEs, as proposed by Chandrasekar *et al* and have shown how their procedure may be brought within the general ambit of the adjoint equation method. In a similar spirit we have derived the coupled adjoint equations for analysis of coupled systems of second-order ODEs. Its use has been illustrated for a system occurring in the context of stellar dynamics. It is obvious that the procedure can easily be extended to systems of higher order equations. Lastly, it may be mentioned that one can apply this method to the equations of the Painlevé–Gambier classification and that this is currently being pursued.

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